Killing-Yano tensors, non-standard supersymmetries and an index theorem

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The existence of Killing-Yano tensors on space-times can be probed by spinning particles. Specifically, Dirac particles possess new fermionic constants of motion corresponding to non-standard supersymmetries on the particle worldline. A geometrical duality connects space-times with Killing-Yano structure, but without torsion, to other space-times with Killing-Yano structure and torsion. A relation between the indices of the Dirac-operators on the dual space-times allows to express the index on the space-time with torsion in terms of that of the space-time without torsion.

It is standard procedure in general relativity to introduce the notion of test particles, idealized as structureless infinitely small mass points, to probe the geometry of space-time by identifying their orbits with geodesics [1]. However, not all the geometric properties of space-time are encoded in the geodesics. Specifically, there can be structures on the space-time related to rotation, which become manifest in the dynamics of spin. Examples of spin-related structures which can live on a space-time are torsion and Killing-Yano tensors.

Representing the particle's worldline co-ordinates at proper time τ by $x^\mu(\tau)$, and the components of its spin by the anti-symmetric tensor of dipole moments $S^{\mu\nu}(\tau)$ (following ref.[2]), the motion of the spinning particle in a torsion-free space-time is described by a generalization of the geodesic equation including coupling of the spin to the background curvature [3]:

$$
\begin{aligned}
\frac{D^2 x^\mu}{D\tau^2} &= \frac{1}{2} S^{\kappa\lambda} R_{\kappa\lambda}{}^\mu{}_\nu \dot{x}^\nu, \\
\frac{DS^{\mu\nu}}{D\tau} &= 0.
\end{aligned}
\tag{1}
$$

Thus the spin-tensor is covariantly constant. The covariant world-line derivative in these equations represents the pull-back of the Riemann-Christoffel connection to the world-line. The standard geodesic equation is reobtained in the limit $S^{\mu\nu} = 0$.

It is not straightforward in general to derive these equations from a variational principle. The way we proceed here is to represent the spin of the particle in terms of anti-commuting c-numbers, the Grassmann-odd co-ordinates $\psi^\mu(\tau)$, as:

$$S^{\mu\nu} = -S^{\nu\mu} = -i\psi^\mu\psi^\nu. \tag{2}$$

In this formalism the equations of motion (1) are obtained directly from the supersymmetric extension of the action for geodesic motion [4, 5]. The price to be paid is of course that the spin variable has no direct physical interpretation; this can be overcome by interpreting it as a symbol for the spin of particles in quantum theory [6], the physical value of the spin-components being obtained by an averaging procedure over all spin-histories along the world-line. Alternatively one may proceed directly to a hamiltonian description. This we will also do, but we will keep the representation (2) of the spin-components as it allows to uncover a remarkable rich structure of symmetries and conservation laws which are obtained only by more cumbersome guesswork otherwise. This guess work can be much simplified by first developing the theory in terms of Grassmann co-ordinates, writing down and solving the equations of motion, and finally omitting all aspects having to do with the Grassmannian construction of $S^{\mu\nu}$, if desired.

In terms of the covariant momentum Π_μ:

$$\Pi_\mu = p_\mu - \frac{1}{2}\,\omega_\mu \cdot S = g_{\mu\nu}\dot{x}^\nu, \tag{3}$$

with p_μ the canonical momentum and ω_μ the spin-connection, the Hamiltonian for the spinning particle is

$$H = \frac{1}{2}\,g^{\mu\nu}\Pi_\mu\Pi_\nu. \tag{4}$$

The evolution of any scalar function on the phase-space is described by the Poisson-Dirac bracket [7]

$$\frac{dA}{d\tau} = \{A, H\},$$

$$\{A, B\} = D_\mu A \frac{\partial B}{\partial \Pi_\mu} - \frac{\partial A}{\partial \Pi_\mu} D_\mu B + R_{\mu\nu}\frac{\partial A}{\partial \Pi_\mu}\frac{\partial B}{\partial \Pi_\mu} + i(-1)^A g^{\mu\nu}\frac{\partial A}{\partial \psi^\mu}\frac{\partial B}{\partial \psi^\nu}, \tag{5}$$

with the covariant derivatives defined in [7]. Constants of motion can be found by requiring scalar phase-space functions to commute with the Hamiltonian in the sense of the brackets (5). A universal constant of motion for the theories discussed here is the supercharge Q:

$$Q = \Pi_\mu\psi^\nu, \qquad \{Q, H\} = 0, \qquad \{Q, Q\} = -2iH. \tag{6}$$

The physical interpretation of this equation is simple: $Q = 0$ is the condition for the time-components of the spin to vanish in the restframe. From (6) this condition is now seen to be compatible with the dynamical equations. Other constants

of motion exist if the space-time admits additional structures like Killing vectors and tensors. Generalizing the construction for spinless particles, in the presence of a Killing vector field $K_\mu(x)$ there is a constant

$$J(x, \Pi, \psi) = K^\mu \Pi_\mu + \frac{1}{2} B_{\mu\nu} S^{\mu\nu}, \tag{7}$$

with $2B_{\mu\nu} = K_{\nu,\mu} - K_{\mu,\nu}$. In fact, this quantity is not only constant, but also superinvariant: $\{J, Q\} = 0$. Similarly, the existence of a symmetric Killing tensor $K_{\mu\nu}$ implies another constant of motion

$$Z(x, \Pi, \psi) = \frac{1}{2} K^{\mu\nu} \Pi_\mu \Pi_\nu - \frac{1}{2} S^{\mu\nu} I_{\mu\nu}^\lambda \Pi_\lambda - \frac{1}{4} S^{\mu\nu} S^{\kappa\lambda} G_{\mu\nu\kappa\lambda}, \tag{8}$$

with the tensors $I_{\mu\nu}^\lambda(x)$ and $G_{\mu\nu\kappa\lambda}(x)$ solutions of the differential equations

$$I_{\mu\nu(\kappa;\lambda)} = R_{\mu\nu\sigma(\kappa} K_{\lambda)}{}^\sigma, \qquad G_{\mu\nu\kappa\lambda;\rho} = R_{\sigma\rho[\mu\nu} I_{\kappa\lambda]}^\sigma. \tag{9}$$

Clearly, in the absence of spin one gets the usual constants of motion associated with Killing vectors and tensors. In the presence of spin there are additional terms reflecting the spin-orbit coupling. Moreover, there can also be constants of motion that exist *only* for spinning particles. We have already discussed the standard supercharge (6); but additional conserved supercharges may exist if the background geometry admits a Killing-Yano tensor $f_{\mu\nu}$ [7]. In such a geometry there exists an additional superinvariant constant of motion $Q_f(x, \Pi, \psi)$ defined by

$$Q_f = f_\mu{}^\nu \Pi_\nu \psi^\mu + \frac{i}{3!} H_{\mu\nu\lambda} \psi^\mu \psi^\nu \psi^\lambda. \tag{10}$$

Here $f^\mu{}_\nu$ are the mixed components of a Killing-Yano 2-form $f = f_{\mu\nu} dx^\mu \wedge dx^\nu$, which by definition posesses a 3-form field strength $H = df$ with the property $H_{\mu\nu\lambda} = f_{[\mu\nu;\lambda]} = f_{\mu\nu;\lambda}$. The superinvariance implies $\{Q_f, Q\} = 0$. The most interesting aspect of the bracket algebra for the new supercharge is the relation

$$\{Q_f, Q_f\} = -2iZ, \tag{11}$$

with Z of the form (8), and

$$K_{\mu\nu} = f_{\mu\lambda} f_\nu{}^\lambda, \qquad G_{\mu\nu\lambda\kappa} = R_{\mu\nu\rho\sigma} f_\lambda{}^\rho f_\kappa{}^\sigma + \tfrac{1}{2} H_{\mu\nu}{}^\sigma H_{\lambda\kappa\sigma},$$

$$I_{\mu\nu\lambda} = f_\mu{}^\sigma f_{\nu\lambda;\sigma} - f_\nu{}^\sigma f_{\mu\lambda;\sigma} + H_{\mu\nu\sigma} f_\lambda{}^\sigma. \tag{12}$$

In the special case that $Z = H$ we have standard $N = 2$ supersymmetry. In all cases with $H \neq 0$ this is impossible, and we have a true non-standard supersymmetry. Examples of such structures can be found e.g. in Kerr-Newman or Taub-NUT space-time [7, 8]. Killing-Yano tensors have been discussed previously in the context of non-standard Dirac operators which can be diagonalized

simultaneously with the standard one [9, 10]. These correspond precisely to the quantum-mechanical version of the supercharges presented above, in the sense of the correspondence relation

$$p_\mu \rightarrow -i\partial_\mu, \qquad \psi^\mu \rightarrow \frac{i}{\sqrt{2}}\,\gamma_5\gamma^\mu, \tag{13}$$

with γ^μ the local version of the Dirac matrices: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. This correspondence leads to the result that modulo a factor $\sqrt{2}$

$$Q \rightarrow \gamma_5\gamma^\mu D_\mu \equiv \gamma_5 \mathcal{D}, \qquad Q_f \rightarrow \gamma_5\gamma^\mu \left(f_\mu{}^\nu D_\nu - \frac{1}{3!}\,\sigma^{\kappa\lambda}H_{\mu\kappa\lambda} \right) \equiv \gamma_5\mathcal{D}_f. \tag{14}$$

Like the pseudo-classical supercharges these Dirac-operators anti-commute; defining $\mathcal{D}_f^5 \equiv \gamma_5\mathcal{D}_f$, it then follows that $\mathcal{D}$ and $\mathcal{D}_f^5$ commute: $[\mathcal{D}, \mathcal{D}_f^5] = 0$. This has some interesting consequences; in particular, the index of these two operators, defined as the difference between left- and righthanded zero-modes: $\Delta = n_+^0 - n_-^0$, is the same. In fact, only the simultaneous zero-modes of these operators can produce a non-vanishing contribution to the trace of γ_5 over all of the physical state space; hence [14]:

$$\mathrm{Tr}\,\gamma_5 = \Delta\,[\mathcal{D}] = \Delta\,[\mathcal{D}_f^5]. \tag{15}$$

To evaluate such traces over infinite-dimensional spaces one has to regularize the expressions (15). A convenient way is to do this in terms of the Witten-index of the corresponding supersymmetric quantum-mechanical model [11]:

$$\Delta\,[\mathcal{D}] = \lim_{\beta\to0}\,\mathrm{Tr}\,\left((-1)^F e^{-\beta H}\right). \tag{16}$$

This expression can be rewritten as a path-integral with the pseudo-classical action given by the hamiltonian (4) of our supersymmetric spinning particle model, using periodic boundary conditions for the fermionic degrees of freedom ψ^μ. Eq.(15) now suggests one should also be able to write this quantity as a regularized trace

$$\Delta\,[\mathcal{D}_f^5] = \lim_{\beta\to0}\,\mathrm{Tr}\,\left((-1)^F e^{-\beta Z}\right). \tag{17}$$

where we have replaced $\mathcal{D}$ by $\mathcal{D}_f^5$, and correspondingly H by the square of the Killing-Yano supercharge Z. This corresponds to a theory in which the roles of Hamiltonian and Killing-tensor have been interchanged, as well as those of the supercharge Q and the Killing-Yano supercharge Q_f. In ref.[12] this dual relation between metrics and Killing tensors, which has been observed independently in [13], and between supercharges and Killing-Yano tensors was investigated systematically. It was shown, that for spinning particles the procedure in general works between the original space (without torsion), and the Killing-dual space only if the latter admits torsion. However, it is then no longer clear if the procedure (17) to compute the trace of γ_5 has the same meaning as previously, as now it refers

to the index of a Dirac operator on a different space-time, with different metric and with torsion, to which is generally added the problem of having to include boundary contributions [15]. On the other hand, if the equality still holds, the procedure can be turned around to express the index of a Dirac operator on some specific space-time with torsion in terms of that of another Dirac operator on a space-time without torsion. Recently Peeters and Waldron have managed to do the computation of the index on the Killing-dual space-times with torsion and non-empty boundary directly [15]; in the specific examples they have checked, they found it to agree with the known result for the original Dirac operator.

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